Assessing Mortgage Servicing Rights Using a Reduced-Form Model: The Effects of Interest Rate Risks, Prepayment and Default Risks, and Random State Variables

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Abstract

Accurately assessing the value of a mortgage servicing rights (MSRs) is an important task for market participants. It is critical for financial institutions to report the fair value of their MSR holdings. It is also useful for those who trade MSRs because a fair value establishes a basis for price negotiation. However in practice, establishing an MSR's fair value is difficult because it involves various revenues and costs. The stochastic property of interest rates and likelihoods of prepayment and default make such estimations particularly challenging. This paper derives a closed-form pricing formula using the reduced-form model for accurately appraising the fair MSR fee. Furthermore, sensitivity of the MSR fair value is explored for the model parameters and examples are given to illustrate the applications of the derived close-form formula for profit analysis and break-even values of the parameters. Actual mortgage data are used to estimate model parameters and conduct numerical analyses. The results suggest that our model can be used by financial institutions and regulators to establish a fair value for MSRs, and in turn better manage risk and project costs.

Keywords: mortgage servicing rights, reduced-form model, closed-form pricing formula, numerical analyses.

1. Introduction

A growing trend throughout the world is the development of mortgage markets in which residential mortgages are consolidated in pools and sold as mortgage-backed-securities (MBSs). The development of MBS markets fosters the need for companion markets—for mortgage servicing rights (MSRs). Mortgage servicers have become increasingly important with the growth of the securitization market, because they are responsible for managing the relationship between the borrower, guarantor, and the investor or trustee of a given loan during the life of the MBS. The mortgage banking industry has experienced tremendous growth in the MSR arena since the 1980s. Nowadays, MSRs are a vital part of many financial institutions' operations, as well as a significant component of their portfolios of available investment opportunities.

Accurately assessing an MSR's value is an important task for financial institutions. Under U.S. accounting rules in place since 1995, banks are supposed to report the value of their MSRs on a fair-market basis, or roughly what they would fetch in a sale. A bank must record a loss whenever it sells MSRs for a price below what is marked on the books. Due to heterogeneity among different servicing portfolios and the lack of an active liquidity market on which MSR prices are quoted, banks are required to have an accurate model for determining an MSR's fair value. Determining these values is important not only for the managers of these institutions, but also for the regulators charged with monitoring their activities. The main purpose of this paper is to provide a reasonable model for accurately appraising MSR values. Our model should not only help financial institutions determine fair MSR values, but also help those who purchase or sell servicing establish fair allowable prices.

The first paper described an MSR valuation model includes a specification of the stochastic discounted cash flow in determining MSR prices.[1] In general, to obtain realistic valuations of the servicing contracts for both fixed- rate mortgages (FRMs) and adjustable-rate mortgages (ARMs),

[1] See McConnell (1976).

researchers need to use a framework that provides for stochastic one-period interest rates, stochastic inflation rates, and the mortgagor's prepayment decision.[2] Nowadays, the option-adjusted valuation approach (also defined as the structure-form approach) is usually used for appraising MSRs because the prepayment and default decisions can be treated as the borrower’s options.[3] In conventional studies, the prepayment model is assumed to be volatility-independent. This assumption implies that mortgagors are indifferent to the volatility of interest rates when deciding if and when to refinance. Several recent studies indicate that the volatility of interest rates can influence the level of prepayments and the value of MSRs.[4] To better determine the true value of MSRs, such scholars employ option-based models that assume volatility-dependent prepayment.[5]

When investigating an MSR’s value, one must first understand the basics of the servicing contract and the cash flow categories for MSRs. These issues have been discussed in several papers, because the value of an MSR depends on the servicing revenue and cost streams.[6] These functions, which servicers are required to perform during the life of the loan, include, but are not limited to, collecting monthly payments (principal, interest, taxes, insurance), remitting the payments to the appropriate parties (principal and interest to the owners of the loans, taxes and insurance to the counties and insurance companies) and foreclosure services, if required. Customarily, the servicer is entitled to collect the monthly fee, as well as other fees such as floating income and late-payment fees. However, servicers also have significant expenses associated with servicing. These include administrative costs as well as delinquency and foreclosure costs.

Mortgage servicers earn revenue from several sources. 1. The most obvious source of income is the servicing fee that borrowers pay. The fee is deducted from the interest portion of the borrower’s monthly mortgage payment. However, the servicer receives this fee only if the borrower

2 See Van Drunen and McConnell (1988).
3 See Brown, Hayre, Lauterbach, Payne, and Zimmerman (1992); Buttimer and Lin (2005); Lin and Ho (2005); Lin and Chu (2006).
4 See Lin (2003); Lin and Ho (2005); Buttimer and Lin (2005); Lin and Chu (2006).
5 See Kalotay and Fu (2008).
6 See Van Drunen and McConnell (1988); Aldrich, Greenberg and Payner (2001).

is making payments. 2. A major source of servicer revenue is the interest income from the escrow account. The servicer earns interest on the homeowner's principal and interest payments from the time they are received until they are forwarded to the appropriate agency or directly to the owner of the mortgage. Moreover, to protect the lending institution's interest payment associated with the mortgage, homeowners are required to pay real estate taxes and insurance premiums into an escrow account. The purpose is to insure prompt payment of these expenses as they come due. Servicers generate income from the interest of these escrow balances. 3. The final source of revenue is late fees. When a borrower does not make a payment by the due date, the servicer is allowed to charge and collect late fees on the unpaid amount.

Servicers' costs typically consist of direct servicing costs and default-related costs. The direct costs include personnel, occupancy and equipment, outsourcing, and miscellaneous expenses associated with servicing the loan. If the loan is defaulted, default-related fees, such as delinquency and foreclosure costs, are generated. If a mortgagor defaults on a proper loan, the servicer will typically have to advance the funds to the agencies. Even though this money is eventually repaid to the servicer when the property is sold, the servicer must bear the financing cost in advance. If the servicer initiates foreclosure of the property, the associated expenses include court costs, legal fees, and any administrative costs.

Our valuation model for calculating a fair MSR fee incorporates all the foregoing types of cash flow (i.e., all the servicing revenues and costs). Compared with traditional studies, many researchers define the MSR value as the total profit generated by an MSR contract. This value is calculated from the gross revenue charged from the borrower minus the servicer's costs associated with providing the services over the life of the loan. In our model, the MSR fee is the fair MSR value; that is, the fee is calculated based on the assumption of a break-event situation, in which the servicing revenue covers only the servicing costs over the life of the loan. As discussed further below, the profit rate for the MSR owner equals the currently assigned MSR fee minus the fair MSR fee calculated from our model.

An important factor influencing the MSR fee is the terminated portion of the mortgage servicing portfolios, which is defined as the ratio of the terminated mortgage amount prior to maturity to the surviving mortgage amount. When the loan is terminated, the servicing cash flow is truncated as well, and the servicing contract has no remaining value. A borrower can terminate a loan in two ways: default or prepayment. Some studies have indicated that these options have a strong impact on income and expense variables, and thus they affect the value of the loan-servicing contract.[7] Several researchers have applied valuation models that include the borrower's prepayment and default,[8] and some of these have been options-based.[9] Thus, we have incorporated these two termination risks in our model.

Because taking account of borrower decisions is important in the valuation of mortgage-related contracts, most researchers are concerned with how the termination probability and the loss given default can be appropriately measured and predicted. Their results indicate that all the changes in the termination probabilities can be influenced by state variables such as interest rates and housing values.[10] Some studies have shown that MSR fees are likewise sensitive to volatility in these interest rates and in housing prices.[11] In addition, some researchers have stated that as interest- rate volatility increases, so do the prepayment amounts, thereby decreasing the value of the MSR.[12] The current literature indicates that MSR values are significantly affected by changes in the interest rate environment determined by these changes in the timeliness of the estimated prepayments.[13] Therefore, in valuating an MSR it is both feasible and important to model the hazard rates for the prepaid and defaulted portions of a mortgage portfolio as multivariate functions with important

[7] See Van Drunen and McConnell (1988); Brown et al. (1992); Hendershott and Villani (1994); Lin and Chu (2006).
[8] See Van Drunen and McConnell (1988); Aldrich, Greenberg, and Payner (2001).
[9] See Buttimer and Lin (2005); Lin and Chu (2006).
[10] See Green and Shoven (1986); Cunninghan and Capone (1990); Schwartz and Torous (1989, 1993); Quigley and Van Order (1990, 1995); Hurt and Felsovalyi (1998); Frye (2000a, 2000b, 2003); Acharya, Bharath, and Srinivasan (2004); Jokivuolle and Peura (2003); Dermine and Carvalho (2006).
[11] See Buttimer and Lin (2005).
[12] See Lin (2003); Lin and Ho (2005); Lin and Chu (2006).
[13] See Donoghue (2006).

correlated random variables. To accurately appraise the MSR fee, we model these two rates as such specifications.

Also requiring consideration are the prepayment and default portions of the mortgage servicing portfolio, as well as the stochastic interest rates, the associated income and costs, and economic factors such as housing prices. As previously mentioned, options-based models have usually been adopted to valuate mortgage servicing contracts. However, when assessing MSR fees with such models, the derivation process involves solving a partial differential equation that is subject to boundary and early termination conditions. This makes it exceedingly difficult to derive a closed-form solution for the MSR fee if the option-based model includes more than two stochastic processes of state variables. Moreover, with this method, it is difficult to identify the critical region of early exercise. Recently, reduced-form models have become more popular for valuating mortgage-related asset prices and analyzing the probabilities of default and prepayment.[14] Compared with options-based models, reduced-form models do not set the boundary conditions and do not require complicated pricing procedures, such as solving complex partial differential equations. Reduced-form models can thereby accurately handle the valuation model includes the multi-dimensional space of correlated state variables. They also make it easier to derive closed-form formulas for the MSR fee. We therefore chose a reduced-form model to valuate the MSR fees.

In summary, an accurate MSR fee strongly depends on reasonable specifications of significant MSR cash flow, the stochastic property of the interest-rate term structure, and the prepayment and default portions of the mortgage servicing portfolio. We incorporate all the above factors in our model to make it more logical without sacrificing generality. The MSR cash flow included in our model consists of income (i.e., the servicing fee, revenue from interest charged from the escrow account, late fees) and costs (direct servicing costs, default-related costs). As mentioned above,

[14] See Kau, Keenan and Smurov (2004); Liao, Tsai, and Chiang (2008); Tsai, Liao, and Chiang (2009); Tsai and Chiang (2012).

changes in the termination probabilities can be influenced by state variables (e.g., interest rates, housing values). To reasonably characterize the termination portions of our model, we therefore specify the prepaid and defaulted hazard rates as multivariate affine functions that include the correlated stochastic state variables.[15] We further derive a general closed-form formula for valuating the MSR fee. To the best of our knowledge, our general pricing model is the first to provide a closed-form formula for the MSR fee using a reduced-form model that simultaneously incorporates so many material factors and relevant variables.

Our model also guides mortgage servicers in recalculating MSR fees so they can protect themselves against losses associated with changing economic conditions. One can conduct numerical analyses to demonstrate the influence of the relevant variables on the MSR fee. For this paper, we used real-world mortgage data to illustrate how one can calculate the termination portion of the mortgage servicing portfolio, and then determine MSR fee based on our derived formula. Also, we provide sensitivity analyses to explore the impact of the model's parameters associated with interest-rate processes and state variables, as well as termination risks and MSR cash flow. This discussion should give readers a better understanding of how the various parameters described above can influence the MSR fee.

It is worth noting that our model and the associated formula can be extended to help mortgage servicers analyze and manage the servicer's profits and losses given the authoritative MSR fee. Second, our formula can be applied to calculate the critical values of the model's parameters (e.g., baseline prepayment, default hazard rates, volatility of the interest rate, housing return). This application should help MSR owners estimate their profits. For example, they can use our formula to calculate the critical value of the baseline prepayment hazard rate given the authoritative MSR fee. If the actual value of the baseline prepayment hazard rate is larger than its critical value, the servicer

[15] The hazard rates are modeled as affine functions of the state variables, as demonstrated in many previous studies (e.g., Lekkas, Quigley, and Van Order, 1993; Jarrow, 2001; Janosi, Jarrow, and Yildirim, 2003; Capone, 2003; Qi and Yang, 2009).

can incur losses. Thus, MSR owners should consider adjusting the authoritative MSR fee. It is important for MSR owners to consider these two issues, but there is no literature to guide them.

The remainder of this paper is organized as follows. Section 2 provides the framework used for valuating the MSR fee. It describes the MSR income and cost components when the borrower pays on schedule, defaults, or prepays. In Section 3, we describe the derivation of the closed-form formula for the MSR fee. We provide an example where the state variables are normally distributed. Section 4 describes sensitivity analyses showing how sensitive the MSR fee is to changes in the model's parameters. We also extend our model to the investigation of mortgage servicers' profits and losses. Section 5 presents a numerical example to demonstrate the application of the model. In the final section, we summarize our findings.

2. The Basic Valuating Framework of Mortgage Servicing Rights

We define the valuation framework as pricing the MSR fee for fully amortized fixed-rate mortgage portfolios. We denote the initial principal balance for servicing these portfolios as $M(0)$. We assign the portfolio coupon rate c with time to maturity at T years. The regular payments from the portfolios are designated as Y. The outstanding unpaid principal balance of the portfolios at time s, $M(s)$, $s \le T$, with no termination risks, is obtained as follows:

$$Y = M(0) \times \frac{c}{1-e^{-cT}} \quad \text{and} \quad M(s) = M(0) \times \frac{1-e^{-c(T-s)}}{1-e^{-cT}}. \qquad (1).$$

Because the terminated portions of the mortgage portfolio can influence the fair pricing of the MSR, we incorporate both the prepayment and default risks in our model. Specifically, we let $P_\theta(u)$ and $P_\pi(u)$ be the prepayment and default portions respectively of the portfolio at each time point u, conditional on survival at time point $u-1$. Then we define the remaining portion of the portfolio until time s, $P_S(s)$, as follows:

$$P_S(s)=\prod_{u=1}^{s}(1-P_\theta(u)-P_\pi(u)). \tag{2}$$

The prepayment and default portions at time s are $P_\theta(s)P_S(s-1)$ and $P_\pi(s)P_S(s-1)$ respectively. Given these specifications, the remaining mortgage balance and payments from the portfolio at time s are $M(s)P_S(s)$ and $YP_S(s)$ respectively. The amounts from the prepaid and defaulted mortgages at time s are $M(s)P_\theta(s)P_S(s-1)$ and $M(s)P_\theta(s)P_S(s-1)$.

As previously mentioned, the critical cash flow servicing components are income (the servicing fee, the interest from the escrow account, the late fee) and expenses (direct servicing costs, default-related costs). Following is a detailed explanation of the specifications for cash flow in our model. First, we give the servicing fee specification. Because it is a fixed percentage of the unpaid mortgage balance, we denote this fee for the per-monetary-unit mortgage balance as ξ. At time s, the revenue from the servicing fees is then $\xi M(s)P_S(s)$. Therefore, at initial time t, the expected current total revenue is equal to the servicing fee for the per-monetary-unit mortgage balance (ξ) times the sum of the expected unpaid balances (RS). That is:

$$\xi RS=\xi E_t[\sum_{s=t}^{T}e^{-\int_t^s r(u)du}M(s)P_S(s)], \tag{3}$$

where $E_t[\cdot]$ is the expectation based on the information available to the investor at time t, and $r(u)$ is the instantaneous default-free short-term interest rate at time u.

Next, we describe the specification regarding the interest income from the escrow account. Servicers earn floating interest from escrow balances (including the amount of taxes, insurance premiums, monthly principal and interest payments, and prepayment balances in interest-bearing accounts) prior to remittance to the owner of the mortgage, taxing authority, and insurance company. For simplicity, we define t_{d_1} as the time period from when the funds are received from the mortgagor until they are remitted to the lender.[16] For example, if the money is put in the escrow account for one week, the deposit period is $t_{d_1} = 1/52$ years. The longer this period, the more interest the servicer receives.

To specify the cash flow, we denote the taxes and insurance premiums for the per-monetary-unit scheduled payments into the escrow account at time s as $\rho(s)$, and the total amounts received there from at time s as $\rho(s)YP_S(s)$. Because the interest rate at time s is $r(s)$ and the deposit period is t_{d_1}, the interest that the servicer earns at time s is $r(s)t_{d_1}\rho(s)YP_S(s)$.

When one borrower prepays a mortgage, the servicer may deposit the repaid amount into the escrow account until it is disbursed to the owner of the mortgage. The servicer then earns the interest from these prepayments. We also define the deposit period (from when the repaid amounts are received until they are forwarded to the owner of the mortgage) as t_{d_1} years. Because the prepayment amount at time s is $M(s)P_\theta(s)P_S(s-1)$, the interest that the servicer receives is $r(s)t_{d_1}M(s)P_\theta(s)P_S(s-1)$.

Although the borrowers have to make monthly principal and/or interest payments, some pay in advance and some make delayed payments. We let $\kappa_1(s)$ and $\kappa_2(s)$ represent the ratio of advance payments and the ratio of delayed payments respectively at time s.[17] If the borrower pays before the due date, the servicer earns the floating interest income from these payments prior to remittance to the owner of the mortgage. We further assume that the length of this period is the same as the deposit time: t_{d_1} years. Because the advance payment is $\kappa_1(s)YP_S(s)$, it produces interest income $r(s)t_{d_1}\kappa_1(s)YP_S(s)$. According to our previous specifications, the total revenue from the balance in

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[16] This assumption can be relaxed. However, the model becomes very complex if many deposit periods are specified.

[17] The ratio of advance payments and the ratio of delayed payments are defined as the advance payments and delayed payments divided by the remaining mortgage payments from the portfolio respectively.

the escrow account is

$$r(s)\, t_{d_1}(\rho(s) YP_S(s) + M(s) P_\theta(s) P_S(s-1) + \kappa_1(s) YP_S(s)) .$$

Therefore, the total expected present value of the revenue from the escrow account during the period from t to T (denoted as RE) can be described as follows:

$$RE = E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} r(s) t_{d_1}(\rho(s) YP_S(s) + \kappa_1(s) YP_S(s) + M(s) P_\theta(s) P_S(s-1))] . \tag{4}$$

If the borrower does not make a payment prior to the due date, the servicer must advance the funds to the owner of the mortgage. Even though the funds are eventually repaid to the servicer, the servicer must bear the opportunity cost of financing the advance. Thus, the servicer is allowed to charge and collect late fees on the unpaid amount. We let the rate of the late fee for the per-monetary-unit unpaid amount be $\varphi(s)$. We assume the reimbursed time for these late payments to be t_{d_2} years. Because the late payment is $\kappa_2(s) YP_S(s)$, the opportunity cost for the servicer is $r(s) t_{d_2} \kappa_2(s) YP_S(s)$, and the servicer's income from the late fees is $\varphi(s)\kappa_2(s) YP_S(s)$. In general, the late fee paid by the borrower is larger than the reimbursed interest received by the lender. Therefore, we define the net late fee that the servicer earns as the late fee minus the opportunity cost. This difference can be calculated from the formula $(\varphi(s) - r(s) t_{d_2})\kappa_2(s) YP_S(s)$. According to the previous specification, we can obtain the total expected present value of the revenue from the net late fee (RL) from times t to T as follows:

$$RL = E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} (\varphi(s) - r(s) t_{d_2})\kappa_2(s) YP_S(s)] . \tag{5}$$

As for the servicing expenses, our model allows direct costs (personnel, occupancy, equipment) and default-related costs (delinquency, foreclosure). The direct cost can be divided into two parts: the fixed cost and the variable cost. We assume the variable cost to come primarily from personnel expenses. We let the direct servicing cost at time s be:

$$\varpi(s) = \varpi_0(s) + \varpi_1(s)N,$$

where $\varpi_0(s)$ is the fixed servicing cost at time s, $\varpi_1(s)$ is the wage for one employee, and N is the number of employees. We obtain the total expected present value of the servicing cost (SC) from time t to T as follows:

$$SC = E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} \varpi(s)] = E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} (\varpi_0(s) + \varpi_1(s)N)]. \tag{6}$$

The servicer is responsible for handling defaulted loans. We divide the default-related costs into two parts: the fixed cost and the variable cost. We assume that the variable cost depends primarily on the length of the period from when the mortgage is defaulted to when the funds are recovered. We let the default-related costs for the per-monetary-unit default mortgage be $\upsilon(s)$. Therefore, we have $\upsilon(s) = \upsilon_0(s) + \upsilon_1(s)t_f$, where $\upsilon_0(s)$ is the constant default-related cost, t_f is the treatment period for the defaulted loan, and $\upsilon_1(s)$ is the marginal rate of the default-related cost with respect to the length of the treatment period. Because the default amount at time s is $M(s)P_\pi(s)P_S(s-1)$, the total default-related costs can be expressed as $\upsilon(s)M(s)P_\pi(s)P_S(s-1)$. We obtain the total expected present value of the default-related costs from time t to T as follows:

$$\begin{aligned} FC &= E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} \upsilon(s)M(s)P_\pi(s)P_S(s-1)] \\ &= E_t[\sum_{s=t}^{T} e^{-\int_t^s r(u)du} (\upsilon_0(s) + \upsilon_1(s)t_f)M(s)P_\pi(s)P_S(s-1)]. \end{aligned} \tag{7}$$

According to the preceding definition, and assuming no arbitrage, we have

$$\xi \times RS + RE + RL = SC + FC.$$

The left side of this equation is the amount that the mortgage servicer expects to receive at the present time, and the right side represents the costs that the mortgage servicer expects to pay out at the present time. Thus, the MSR fee can be obtained as follows:

$$\xi = RS^{-1}[SC + FC - RE - RL]. \quad (8)$$

This derived MSR fee is a break-even fee, which means that the servicing revenue covers only the direct servicing costs at the time the servicer requires the borrower to pay servicing fee ξ. If the servicing fee the borrower actually pays is larger than ξ, the owner of the MSR makes a profit. Otherwise, the owner takes a loss.

3. An Application Case: Deriving the Closed-Form Formula for an MSR Fee Using the Intensity Model under the Assumption of a Normal Distribution

We assume that the valuation framework for deriving the closed-form solution of the MSR fee is continuous in time. When using the reduced-form model to assess the MSR fee, the terminated portions of the mortgage portfolio need to be specified. We let $\theta(s)$ and $\pi(s)$ be the hazard rates for the prepaid and defaulted portions respectively at time s. Therefore, the surviving portion of the portfolio at time s can be described as follows:

$$P_S(s) = \exp(-\int_t^s (\theta(u) + \pi(u))du). \quad (9)$$

Consequently, the prepayment and default portions at time s can be obtained as follows:

$$P_\theta(s) = \theta(s)P_S(s), \text{ and } P_\pi(s) = \pi(s)P_S(s). \quad (10)$$

Research has shown that interest rates and housing prices significantly affect borrowers' prepayment and default decisions.[18] We chose these two variables to be the main sources of uncertainty in determining the termination rates in our model. The vector of the state variables is denoted as $W(s) = [r(s), \quad r_H(s)]'$, where $r_H(s)$ represents the housing return. As demonstrated in many previous studies,[19] the hazard rates for the prepayment and default portions are multivariate affine functions:

[18] See Yang, Buist and Megbolugbe (1998); Clapp, Goldberg, Harding and LaCour-Little (2001); Azevedo-Pereira, Newton, and Paxson (2003).

[19] See Lekkas, Quigley, and Van Order (1993); Jarrow (2001); Janosi, Jarrow, and Yildirim (2003); Capone (2003); Qi and Yang (2009).

$$l_i(s) = a_0^l(i) + A_l W(s), \text{ for } l = \theta, \pi, \tag{11}$$

where $a_0^l(i)$ is the deterministic baseline hazard rate for the prepayment and default portions (given $l = \theta, \pi$ respectively) for a mortgage issued after i years, and $A_l = [a_r^l \quad a_1^l]'$ is a vector of coefficients in which a_r^l and a_1^l denote the relative magnitudes of the interest rate effect and the housing return effect respectively on the hazard rates for the prepayment and default portions.

Based on the above specifications, we have:

$$e^{-\int_t^s r(u)du} P_S(s) = \exp(-\int_t^s (r(u) + \theta(u) + \pi(u))du) = \exp(-A_B^0(s) - A_E' X(s)), \tag{12}$$

where

$$A_B^0(s) = A_\theta^0(s) + A_\pi^0(s); \quad A_l^0(s) = \int_0^{s-t} a_0^l(u)du, \text{ for } l = \theta, \pi; \text{ and } A_E = [1 + a_r^\theta + a_r^\pi \quad a_1^\theta + a_1^\pi]'.$$

For simplicity, we let the following parameters have constant values: the tax and the insurance fee, the ratio of advance payments, the ratio of delayed payments, the late fee, the number of employees, the treatment period for the defaulted loan, and the fixed costs and the marginal variable costs (i.e., direct servicing, default-related). In addition, we let the structure of the interest rate be flat. We can rewrite Equations (3)-(7) as the following:

$$RS = \int_t^T M(s) E_t[\exp(-A_B^0(s) - A_E' X(s))]ds; \tag{13}$$

$$RL = \int_t^T \varphi \kappa_2 Y E_t[\exp(-A_B^0(s) - A_E' X(s)]ds - \int_t^T \kappa_2 Y t_e E_t[r(s)\exp(-A_B^0(s) - A_E' X(s)]ds; \tag{14}$$

$$RE = \int_t^T t_e \rho(s) Y E_t[r(s)\exp(-A_B^0(s) - A_E' X(s)]ds + \int_t^T t_e \kappa_1(s) Y E_t[r(s)\exp(-A_B^0(s) - A_E' X(s))]ds$$

$$+ \int_t^T t_e M(s) E_t[r(s)\theta(s)\exp(-A_B^0(s) - A_E' X(s)]ds; \tag{15}$$

$$SC = \int_t^T (\varpi_0(s) + \varpi_1(s)N) E_t[e^{-\int_t^s r(u)du}]ds = (\varpi_0 + \varpi_1 N)\int_t^T E_t[e^{-\int_t^s r(u)du}]ds; \text{ and} \tag{16}$$

$$FC = (\upsilon_0 + \upsilon_1 t_f) \int_t^T M(s) E_t[\pi(s) \exp(-A_B^0(s) - A_E' X(s)] ds \, . \tag{17}$$

To solve each three expected values $E_t[\cdot]$ in the explicit formula is the key points for obtaining the closed-form formula of the MSR fee. Previous researchers have usually assumed that the stochastic processes of the state variables in valuation models are normally distributed,[20] and we make this assumption as well. We assume that the interest rate and the housing return follow the Vasicek model (Vasicek, 1977; Heath, Jarrow, and Morton, 1992).[21] The dynamic behavior processes of the default-free short-term interest rate $r(t)$ and the housing return $r_H(t)$ are expressed as follows:

$$dr(t) = a(\bar{r}(t) - r(t))dt + \sigma_r dZ_r(t) \text{; and}$$

$$dr_H(t) = a_H(\bar{r}_H(t) - r_H(t))dt + \sigma_H dZ_H(t) \, ,$$

where $\bar{r}(t)$ and $\bar{r}_H(t)$ are the long-term interest rate and the housing return, deterministic functions of t; a and a_H are the speed of adjustment for the short-term interest rate and the housing return, a positive constant; σ_r and σ_H are the volatilities of the short-term interest rate and the housing return, also a positive constant; and $Z_r(t)$ and $Z_H(t)$ represent the standard Brownian motions of the short-term interest rate and the housing return respectively. Let ϕ_{rH} , a constant value, denote the correlation between $dZ_r(t)$ and $dZ_H(t)$.

We denote the mean of $W(s)$—the vector of the state variables—as $\mu_W(s)$; the mean of $X(s)$ —the vector of the cumulative state variables—as $\mu_X(s)$ with $X(s) = \int_t^s W(u)du$; the variance of $W(s)$ as $V_W(s)$; the variance of $X(s)$ as $V_X(s)$; and the covariance of $W(s)$ and $X(s)$ as $V_{X,W}(s)$.[22] Because we assume that $W(s)$ and $X(s)$ are normally distributed, we have (Tsai and Chiang, 2011):

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[20] See Chen and Yang (1995); Schwartz and Torous (1992); Tsai and Chiang (2012).

[21] Research has shown that the Vasicek model perform well in pricing mortgage-backed securities (Chen and Yang, 1995).

[22] The derivations of their expected value and the variance-covariance matrix are shown in Appendix A.

$$E_t[e^{-\int_t^s r(u)du}] = B(s) = \exp(-\mu_R(s) + \frac{1}{2}V_R(s))$$, is the value of a zero coupon bond; (18)

$$E_t[\exp(-A_B^0(s) - A_E' X(s)] = \Phi(s); \tag{19}$$

$$E_t[r(s)\exp(-A_B^0(s) - A_E' X(s)] = \Phi(s)\Phi^r(s); \tag{20}$$

$$E_t^Q[\pi(s)\exp(-A_B^0(s) - A_E' X(s)] = \Phi(s)\Phi^\pi(s);$$ and (21)

$$E_t^Q[r(s)\theta(s)\exp(-A_B^0(s) - A_E' X(s)] = \Phi(s)\ \Phi^{r,\theta}(s); \tag{22}$$

where

$$\Phi(s) = \exp(-A_B^0(s) - A_E\mu_X(s) + \frac{1}{2}A_E'V_X(s)A_E);\ \ \Phi^\pi(s) = a_0^\pi(s) + A_\pi'\mu_W(s) - A_E'V_{XW}(s)A_\pi;$$

$\Phi^r(s) = A_r'\mu_W(s) - A_E'V_{XW}(s)A_r;\ \ A_r = [1 \quad 0]'$;. and

$$\Phi^{r,\theta}(s) = (A_r'\mu_W(s) - A_E'V_{XW}(s)A_r)\ (a_0^\theta + A_\theta'\mu_W(s) - A_E'V_{XW}(s)A_\theta) + A_r'V_W(s)A_\theta.$$

These specifications allow us to derive the closed-form formula for the MSR fee as follows:

$$\xi = (\int_t^T M(s)\Phi(s)ds)^{-1} \times (\int_t^T ((\varpi_0 + \varpi_1 N)B(s) + (v_0 + v_1 t_f)\Phi^\pi(s)M(s)\Phi(s))ds)$$

$$-\int_t^T ((\rho+\kappa_1)Yt_{d_1}\Phi^r(s) + \Phi^{r,\theta}(s)t_{d_1}M(s) + (\varphi\kappa_2 Y - \kappa_2 Yt_{d_2}\Phi^r(s)))\Phi(s)ds). \tag{23}$$

In general, MSR fees are assigned by the government. If we let this assigned fee be $\bar{\xi}$, the profit earned by the owner of the MSR can be expressed as $\pi_{MSR} = \bar{\xi} - \xi$. In other words, if ξ increases (decreases), the servicer's profit consequently decreases (increases).

4. Sensitivity Analyses: The Influences of the Model's Parameters on the MSR Fee

Our closed-form formula should help mortgage servicers recalculate an accurate MSR fee and manage their MSR risk in response to changing economic conditions. In this section, we provide steady state analyses of these model's parameters aimed at investigating how the relevant variables influence the MSR fee based on our derived formula. According to our model, the MSR fee is influenced by the state variables ($\mu_W(s)$, $\mu_X(s)$, $V_W(s)$, $V_X(s)$, $V_{X,W}(s)$), termination hazard functions (a_0^θ, a_0^π, A_θ, A_π), and the major revenue streams and costs associated with the contracts (ρ, κ_1, κ_2, t_{d_1}, t_{d_2}, φ, ϖ_0, ϖ_1, N, v_0, v_1, t_f). As shown in Appendix A, the state variable should include the initial yield curve $f(t)$, the term-structure evolution parameters (a, σ_r), the housing return evolution parameters (a_H, $\bar{r}_H(t)$, σ_H), and the correlation parameter (ϕ_{rH}).[23] The partial derivative of the MSR fee with respect to the model's parameters is shown in Appendix B.

We first present the results that can be directly interpreted as indicating the impact of each parameter on the MSR fee. The effects of the parameters on the cash flow associated with the MSR fee can be determined as follows:

$\frac{\partial \xi}{\partial \rho} < 0$, $\frac{\partial \xi}{\partial \kappa_1} < 0$, $\frac{\partial \xi}{\partial t_{d_1}} < 0$, $\frac{\partial \xi}{\partial \kappa_2} < 0$, $\frac{\partial \xi}{\partial \varphi} < 0$, $\frac{\partial \xi}{\partial t_{d_2}} > 0$, $\frac{\partial \xi}{\partial \varpi_0} > 0$, $\frac{\partial \xi}{\partial \varpi_1} > 0$, $\frac{\partial \xi}{\partial N} > 0$, $\frac{\partial \xi}{\partial v_0} > 0$, $\frac{\partial \xi}{\partial v_1} > 0$, and $\frac{\partial \xi}{\partial t_f} > 0$.

The above results provide several pieces of information: changes in the tax rate and insurance premiums (ρ), the deposit period (t_{d_1}) (i.e. how long the funds are in the escrow account), the ratio of advance payments (κ_1), the ratio of delayed payments (κ_2) and the late fee (φ). All these parameters decrease the MSR fee; to the contrary, changes in the time at which the late payments are reimbursed (t_{d_2}), the fixed direct servicing cost (ϖ_0), the direct servicing costs of paying employee

[23] Detailed specifications for the interest rate evolution and the housing return evolution are provided in Appendix A. Here we define the symbols as follows: $f(t)$ is the instantaneous forward rate, $\bar{r}_H(t)$ is the long-term housing return, σ_r and σ_H are the volatilities of the short-term interest rate and the housing returns respectively, a and a_H are the adjustment speeds for the short-term interest rate and the housing return respectively, and ϕ_{rH} is the correlation between the interest rate and the housing return.

salaries (ϖ_1), the number of employees (N), the constant default-related cost (υ_0), the marginal rate of the default-related cost (υ_1), and the treatment period for the defaulted loan (t_f) all increase the MSR fee. Thus, we can conclude that an increase in the borrowers' payments and an increase in the deposit period probably raise servicers' revenue, leading to a decrease in the MSR fee. Moreover, increases in servicing costs (direct and default-related), the time at which these late payments are reimbursed, and the treatment period for the defaulted loan increase the servicers' expenses and the risk of them taking a loss. These situations lead to an increase in the MSR fee.

Regarding the influence of termination rates on the MSR fee, we have $\frac{\partial \xi}{\partial a_0^\theta} > 0$ and $\frac{\partial \xi}{\partial a_0^\pi} > 0$. These results show that the baseline hazard rates for the prepayment and default portions of the portfolio positively influence the MSR fee. This is because when prepayments and defaults increase, the income generated from servicing the portfolio decreases and the default-related costs increase. Servicers thus should raise the MSR fee to offset the losses they would otherwise incur. The partial derivatives in Appendix C do not allow one to directly judge whether the impact of the parameters regarding the specifications of the state variables on the MSR fee is positive or negative. In the next section, we use a numerical analysis to explain how they influence the MSR fee.

It is worthy of note that our model can be extended to investigate how mortgage servicers manage their profits and losses. Because the MSR fee is determined by the interest rate, the hazard rates for the terminated portions of the portfolio, the state variables, and the major servicing revenues and expenses, we can represent the MSR fee as follows:

$$\xi = \xi(\Omega) \quad \text{and} \quad \Delta\xi = \sum \frac{\partial \xi}{\partial \Omega} \Delta\Omega ,$$

where Ω represents the previously mentioned parameters for servicing the mortgage portfolio. According to the previous functions, the net profit is $\pi_{MSR} = \bar{\xi} - \xi$, where $\bar{\xi}$ is the authoritative MSR fee. The total profit from the MSR can be defined as $RS \times \pi_{MSR}$. Furthermore, we have

$\Delta\pi_{MSR} = -\Delta\xi$, which means that if the parameters change, the total change in the profit of the MSR holder is $RS \times \Delta\pi_{MSR}$. By using our formula, MSR owners can quickly make accurate estimates of their profits and losses under a variety of economic circumstances. The numerical analyses in the next section address the expected profits of the servicer.

Finally, MSR holders should be concerned about how likely it is that a given economic condition would result in a loss. By using our formula, they can calculate the critical values of the model's parameters (e.g., baseline prepayment and default hazard rates, interest rate volatility, housing return volatility) in a break-even situation. For example, an MRS manager can calculate the critical value for the baseline hazard rate for the prepayment portion of the portfolio. If the actual value is larger than this critical value, the servicer could incur a loss. To obtain the critical values for these important parameters, we let $\bar{\xi} = \xi(\Omega)$; we can then find the inverse of this relationship, $\Omega^* = \xi^{-1}(\bar{\xi})$. The following section gives an example to illustrate this application.

5. Numerical Results

We begin our numerical analyses by presenting our closed-form formula, for which we use market data to estimate the critical parameter values in our model. We choose the short-term interest rate and the housing return as our state variables. The housing prices were taken from the all-transactions indexes obtained through the Federal Housing Finance Agency.[24] These data consist of estimated sale prices and appraisals. Data samples were taken quarterly from January 1998 to April 2010. For the short-term interest rate we use the interest rates for a 3-month U.S. treasury bill. These data were sampled monthly from January 1998 to December 2010. The data pertaining to prepayment and default probabilities were obtained from HUD's 2010 annual report.[25] These data were sampled yearly from 1998 to 2010 (forecasted value in 2010). These prepayment and default probabilities are used to define the corresponding portions of our mortgage servicing portfolios.

[24] The website is http://www.fhfa.gov/Default.aspx?Page=87.

[25] The title of the report is "Actuarial Review of the Federal Housing Administration Mutual Mortgage Insurance Fund (Excluding HECMs) for Fiscal Year 2010." HUD's website is http://portal,hud.gov/hudportal/HUD.

The hazard rates for the prepayment and default portions are calculated from the prepayment and default probabilities taken from the real data.[26] The estimation formulas can be expressed as follows:

$$\theta(t) = \frac{P_\theta(t)}{1 - P_\theta(t) - P_\pi(t)} \text{; and } \pi(t) = \frac{P_\pi(t)}{1 - P_\theta(t) - P_\pi(t)}.$$

We use the quarterly data to estimate the parameters of the short-term interest rate and housing return processes, employing the maximum likelihood method.[27] The estimated results for the dynamic behavior processes of the state variables are shown in Table 1. According to our specifications for the processes of the interest rate and housing return following an O-U process, the estimated parameter values are: a = 0.0714, σ_r = 0.0148, $\bar{r}_H$ = 0.0092, a_H = 0.9, σ_H = 0.0163, and ϕ_{rH} = -0.0575. Because the hazard rates of prepayment and default portions are specified to be multivariate affine functions, that is, $l_i(t) = a_0^l(i) + A_l W(t)$, we estimate the linear parameters (A_l, denoted as $\hat{A}_l$) for the hazard rate functions using the standard least squares method. In practical applications, the expected hazard rates for prepayment and default portions should take only positive values. Thus, to ensure that the forecasted values of $l_i(t)$ are greater than zero, we use an estimation method to obtain reasonable values. Specifically, we let $l_i^e(t) = \vartheta \times l_l(t)$, where $\vartheta < 1$. The maximum value of ϑ is obtained when all the forecasted hazard rates are greater than zero.[28] Consequently, the estimated value for A_l can be obtained as follows:

$$\hat{A}_l = \frac{\sum (l_i^e(t) - \bar{l}_i^e(t))(W(t) - \overline{W}(t))}{\sum (W(t) - \overline{W}(t))^2},$$

[26] The estimated hazard rates need to be martingale hazard rates, which means that the hazard rates are under martingale probability, in this paper. The hazard rates of prepayment and default are defined as physical hazard rates if they are calculated based on the realized data. The physical and martingale hazard rates are equivalent under the assumption of a well-diversifiable portfolio (Jarrow, Lando and Yu, 2005).

[27] To maintain consistency of reporting, data for the quarterly interest rate were obtained by averaging the three monthly interest rates for each quarter.

[28] According to this specification, the state variables can only partly explain the change in the hazard rates. In other words, the portion $(1-\vartheta) \times l_l(t)$ cannot be explained by the state variables. If the value of ϑ is decreased, the influence of the state variables on the hazard rates is reduced.

where $\bar{l}_i^e(t) = \frac{\sum_{i=1}^{n} l_i^e(t)}{n}$; and $\overline{W}(t) = \frac{\sum_{t=1}^{T} W(t)}{T}$.

After obtaining the estimate of $\hat{A}_l$, the estimated baseline hazard rates for prepayment and default portions for each period prior to maturity can be derived as follows:

$$\hat{a}_0^l(i) = \bar{l}_i - \hat{A}_l \overline{W}(t)\text{, where, } \bar{l}_i = \frac{\sum_{t=1}^{T} l_i(t)}{T}.$$

In our example, the estimated values of ϑ for the prepayment and default are 1 and 0.0457 respectively. The estimated parameters for the affine functions are shown in Table 2, where it can be seen that the effects of the relative magnitudes of the interest rate and the housing return on the prepayment and default hazard rates are $A_\theta = [-1.2273 \quad -0.3463]$ and $A_\pi = [-0.0055 \quad -0.0045]$ respectively.[29] The baseline hazard rates of prepayment and default portions, $a_0^\theta(i)$ and $a_0^\theta(i)$, for each period are also shown in Table 2. For simplicity, we assume that the baseline hazard rates of prepayment and default portions are all constants. In other words, $a_0^\theta = \sum_{i=1}^{30} a_0^\theta(i)/30 = 0.2714$ and $a_0^\pi = \sum_{i=1}^{30} a_0^\pi(i)/30 = 0.0140$.

To illustrate how servicers can use our model to investigate their profits and losses from MSRs, we first assign the following basic parameters: $M_0 = \$100$ million, $c = 5\%$, $T = 30$ years; the initial values of $f(t)$ and $r_H(t)$ are 4% and 1% respectively. To operationalize the model, the other parameter values assigned are as follows: $t_{d_1} = 1/52$ years (deposit time), $t_{d_2} = 1/52$ years (reimbursement time for delayed payments), $\rho = 0.5\%$ (tax rate and insurance premiums), $\kappa_1 = 20\%$ (the ratio of advance payments), $\kappa_2 = 10\%$ (the ratio of delayed payments), $\varphi(s) = 1\%$ (rate for the late fee), $\varpi_0 = \$0.03$ Million (fixed direct servicing cost), $\varpi_1 = \$0.001$

[29] For consistency of reporting, the yearly housing return was obtained by summing the quarterly housing returns for each year.

Million (employee's wage), $N = 2$ (number of employees), $\upsilon_0 = \$0.002$ Million , $\upsilon_1 = \$0.0001$ Million (marginal rate for default-related costs), and $t_f = 0.5$ years (treatment period for the defaulted loan). Given these parameters and the estimated parameters previously derived from the model, the MSR fee can easily be calculated using our pricing formula. For example, when the parameters estimated above are plugged into Equation (23), we get an MSR fee (ξ) of 0.1545%, given a mortgage balance of \$100 million. In addition, we have $RS = \$343.424$ Million.

It is worthy of note that servicers can use our closed-form formula to gain an understanding of how the various relevant variables influence the MSR fee. When the above-assigned parameters and the parameters estimated from the model are used in obtaining the partial derivative of the MSR fee (see Appendix B for the derivation), we get the following result:

$\frac{\partial \xi}{\partial \rho} = -\,5.5407 \times 10^{-5}$, $\frac{\partial \xi}{\partial \kappa_1} = -\,5.5407 \times 10^{-5}$, $\frac{\partial \xi}{\partial t_{d_1}} = -\,1.0185 \times 10^{-3}$, $\frac{\partial \xi}{\partial \kappa_2} = -\,6.3252 \times 10^{-4}$, $\frac{\partial \xi}{\partial t_{d_2}} = 2.8812 \times 10^{-4}$, $\frac{\partial \xi}{\partial \varphi} = -\,0.0068793$, $\frac{\partial \xi}{\partial \varpi_0} = 0.050871$, $\frac{\partial \xi}{\partial \varpi_1} = 0.10174$, $\frac{\partial \xi}{\partial N} = 5.0871 \times 10^{-5}$, $\frac{\partial \xi}{\partial v_0} = 1.4672 \times 10^{-4}$, $\frac{\partial \xi}{\partial v_1} = 7.3363 \times 10^{-5}$, and $\frac{\partial \xi}{\partial t_f} = 1.4673 \times 10^{-8}$.

We have now proven that increases in ρ, φ , t_{d_1}, κ_1, and κ_2 reduce the MSR fee, and that increases in ϖ_0, ϖ_1, N , υ_0, υ_1, t_{d_2}, and t_f raise the MSR fee. These findings are also mentioned in our steady-state analysis in Section 4. Moreover, we find that changes in the MSR fee depend primarily on the parameter values for the fixed servicing costs and the wage for one employee (ϖ_0 and ϖ_1).

For the effect of the parameters for the state variables and the hazard rates of prepayment and default portions on the MSR fee, we get the following result: $\frac{\partial \xi}{\partial a_0^{\theta}} = 0.0043647$, $\frac{\partial \xi}{\partial a_0^{\pi}} = 0.0051876$, $\frac{\partial \xi}{\partial f(t)} = -\,0.024180$, $\frac{\partial \xi}{\partial a} = -\,0.0030191$, $\frac{\partial \xi}{\partial \sigma_r} = 0.021406$, $\frac{\partial \xi}{\partial \bar{r}_H(t)} = -\,0.0011312$,

$\frac{\partial \xi}{\partial a_H}$=4.0956$\times 10^{-7}$, $\frac{\partial \xi}{\partial \sigma_H} = -9.7650\times 10^{-6}$, and $\frac{\partial \xi}{\partial \phi_{r,H}}$=8.4712$\times 10^{-7}$. This result shows us that the baseline hazard rates of prepayment and default portions (a_0^{θ}, a_0^{π}), the volatility of short-term interest rates (σ_r), the adjustment speed for the housing return (a_H), and the correlation between the interest rate and the housing return (ϕ_{rH}) all affect the MSR fee in the positive direction. On the other hand, the volatility of the housing returns (σ_H), the instantaneous forward rate ($f(t)$), the adjustment speed of the short-term interest rate (a), and the long-term housing return ($\bar{r}_H(t)$) all affect the MSR fee in the negative direction. Also, the results reveal that the change in the MSR fee depends primarily on the forward rate $f(t)$ and the volatility of the interest rate σ_r. We can then infer that increases in a_0^{θ}, a_0^{π}, σ_r, a_H, and ϕ_{rH} increase the MSR fee, and increases in $f(t)$, a, σ_H, and $\bar{r}_H(t)$ reduce it.

Furthermore, servicers can perform their own profit analyses by using the formula given in Section 4 (i.e., $\Delta\pi_{MSR} = -\Delta\xi$) to obtain the changes of profit for the per-monetary-unit mortgage balance. In our example using the above-estimated parameter values, we have:

$$\begin{aligned}\Delta\pi_{MSR} =& (5.5407\times 10^{-5})\,\Delta\rho + (5.5407\times 10^{-5})\,\Delta\kappa_1 + (6.3252\times 10^{-4})\,\Delta\kappa_2 \\ & + (2.8812\times 10^{-4})\,\Delta t_{d_1} + 0.0068793\,\Delta\varphi - (2.8812\times 10^{-4})\,\Delta t_{d_2} - 0.050871\,\Delta\varpi_0 \\ & - 0.10174\,\Delta\varpi_1 - (5.0871\times 10^{-5})\,\Delta N - (1.4672\times 10^{-4})\,\Delta v_0 \\ & - (7.3363\times 10^{-5})\,\Delta v_1 - (1.4673\times 10^{-8})\,\Delta t_f\,.\end{aligned}$$

It can now be seen that the change in an MSR owner's profit depends primarily on the direct servicing costs (ϖ_0 and ϖ_1). Thus, managers of MSR portfolios are advised to focus their attention on controlling these factors. For example, if the fixed direct servicing cost ϖ_0 increases \$0.01 Million, the MSR owner's total profit drops \$0.1747 Million (i.e., $RS\times\Delta\pi_{MSR}$ =343.424

$\times 0.00050871$).

Finally, as mentioned above, our model can be used to calculate estimates of the critical values for the parameters in a break-even situation. For example, if we assign the MSR fee $(\bar{\xi})$ a value of 0.25%, the estimated critical value for the baseline hazard rate of prepayment is 0.4822. If the baseline hazard rate exceeds the critical value 0.4822, the holder incurs a loss. Thus, MSR owners who adopt our analyses can effectively manage their risks, undertake optimal cost management procedures, and decide whether to adjust their authoritative MSR fee.

6. Conclusion

MSRs are contractual rights exchanged for compensation. They are intended to fulfill a variety of functions, such as collecting monthly payments and remitting tax payments and payments for insurance and foreclosure services. In exchange for performing these functions, the servicer receives a servicing fee. Finding the best way to determine a fair MSR fee is essential for both servicers and borrowers. If the fee set by a servicing company is higher than the borrower should pay, the borrower may be unwilling to contract for the company's services. In contrast, if the calculated premium is lower than what the borrower should pay, the servicer could incur a loss. Therefore, determining an accurate and fair MSR fee is an important consideration when researching how MSR owners can manage their risks. The main purpose of this paper has been to provide a practical formula for determining a fair MSR fee using a reasonable model.

However, calculating such MSR fees can be quite complicated, because changes in the interest rate, as well as the borrower's prepayment and default decisions, create uncertainty in the mind of the MSR owner regarding the expected revenues and expenses. Previous studies have demonstrated that the general risks associated with state variables are important factors affecting the termination probability of a mortgage. To calculate an appropriate MSR fee, these state variables should be incorporated in the valuation model. To reasonably model the effect of termination risks on the MSR fee, we defined the hazard rates for the prepayment and default portion of the portfolio as the affine functions of correlated random state variables. Using a reduced-from model, we were able to overcome the difficulty in deriving a general closed-form formula for calculating the MSR fee. To the best of our knowledge, we are the first to provide such a MSR pricing formula as part of a model that includes such sophisticated specifications. To illustrate how one can use our valuation framework to determine an MSR fee, we gave an example assuming that all the state variables follow a normal distribution.

Our formula should help MSR owners undertake effective risk management, because they can use it to investigate how relevant variables influence the MSR fee. This topic has not been thoroughly addressed in the previous literature. To remedy this situation, we provided sensitivity analyses and provided a numerical example to illustrate the effects of the critical model's parameters on the MSR fee. We employed real mortgage data to illustrate how one can estimate the parameters in the multivariate affine functions regarding the hazard rates of prepayment and default portions; and then use our closed-form formula to calculate the MSR fee. In our numerical example, we found that the MSR fee should be 0.1545% given a mortgage balance of $100 Million.

Our sensitivity analyses show that the MSR fee is negatively affected by the following factors: the tax rate and insurance premiums (ρ), the period of time that the funds are deposited in the escrow account (t_{d_1}), the ratio of advance payments (κ_1), the ratio of delayed payments (κ_2), the late fee rate (φ), the volatility of the housing return (σ_H), the instantaneous forward rate ($f(t)$), the adjustment speed for the short-term interest rate (a), and the long-term housing return ($\bar{r}_H(t)$). To the contrary, the following factors were shown to positively influence the MSR fee: the reimbursement time for the late payments (t_{d_2}), the fixed direct servicing costs (ϖ_0), the direct servicing costs representing employee wages (ϖ_1) and the number of employees (N), the constant default-related costs (υ_0), the marginal rate for the default-related costs (υ_1), the treatment period for the defaulted loans (t_f), the baseline hazard rates for prepayment and default (a_0^{θ}, a_0^{π}), the volatility of the short-term interest rate and housing return (σ_r), the adjustment speed for the housing return (a_H), and the correlation between the interest rate and the housing return (ϕ_{rH}). The change in the MSR fee was shown to depend mainly on the direct servicing costs (ϖ_0 and ϖ_1), the forward rate ($f(t)$), and the volatility of the interest rate (σ_r).

Our profit and loss analyses reveal that ϖ_0 and ϖ_1 are the main factors affecting the MSR fee. Thus, managers are advised to focus their attention on controlling these two factors. Finally, managers can use our model to adjust their management strategies by estimating the critical values of the parameters in our model. For example, if the authoritative MSR fee is 0.25%, the estimated critical value for the baseline hazard rate for the prepayment is 0.4822. In other words, the MRS holder reaps a profit if this rate is less than 0.4822. However, if the rate exceeds this value, the MSR owner incurs a loss. This information alerts the MRS holder of the need to prepare a new management strategy. In short, our pricing formula should make it easier for MSR holders to determine a fair MSR fee and effectively undertake risk management and cost management.

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Table 1: The estimated parameter values for the dynamic behavior processes of the state variables

	a	σ_r	$\bar{r}_H$	a_H	σ_H	$\phi_{r,H}$
Parameters	0.0714	0.0148	0.0092	0.9000	0.0163	-0.0575
P-value	0.2059	0.0000	0.0074	0.0002	0.0000	0.7648

Note: The state variables are the default-free short-term interest rate and the housing return. $\bar{r}_H$ is the housing return, a and a_H are the adjustment speeds for the short-term interest rate and the housing return, σ_r and σ_H are the volatilities of the short-term interest rate and the housing return, and ϕ_{rH} is the correlation between the short-term interest rate and the housing price.

Table 2: The estimated parameters values for prepayment and default hazard rates

	a_r	a_1	$a_0(1)$	$a_0(2)$	$a_0(3)$	$a_0(4)$	$a_0(5)$	$a_0(6)$	$a_0(7)$	$a_0(8)$	$a_0(9)$	$a_0(10)$	$a_0(11)$	$a_0(12)$	$a_0(13)$	$a_0(14)$
Prepayment	-1.2273	-0.3463	0.0991	0.2029	0.2664	0.2869	0.3044	0.3049	0.2993	0.2988	0.2931	0.2988	0.2982	0.2899	0.2762	0.2665
Default	-0.0055	-0.0045	0.0008	0.0085	0.0245	0.0339	0.0365	0.0355	0.0325	0.0287	0.0254	0.0224	0.0197	0.0169	0.0152	0.0136

Note: The mean prepayment hazard rate is 0.2714, and the mean default hazard rate is 0.014. $a_0(t)$ stands for the baseline hazard rates for the prepaid and defaulted portions of the mortgage portfolio for each period. a_r and a_1 are the coefficients of the interest rate and housing return for the hazard rates of the prepaid and defaulted portions of the portfolio.

Table 2: Parameter estimates of the prepayment and default hazard rates (continues)

	$a_0(15)$	$a_0(16)$	$a_0(17)$	$a_0(18)$	$a_0(19)$	$a_0(20)$	$a_0(21)$	$a_0(22)$	$a_0(23)$	$a_0(24)$	$a_0(25)$	$a_0(26)$	$a_0(27)$	$a_0(28)$	$a_0(29)$	$a_0(30)$
Prepayment	0.2700	0.3473	0.2565	0.2445	0.2334	0.2302	0.2520	0.2231	0.2271	0.2238	0.2482	0.2928	0.2462	0.2620	0.2657	0.5349
Default	0.0115	0.0116	0.0103	0.0091	0.0077	0.0065	0.0066	0.0061	0.0059	0.0054	0.0043	0.0043	0.0041	0.0046	0.0019	0.0052

Note: The mean prepayment hazard rate is 0.2714, and the mean default hazard rate is 0.014. $a_0(t)$ stands for the baseline hazard rates for the prepaid and defaulted portions of the mortgage portfolio for each period. a_r and a_1 are the coefficients of the interest rate and housing return for the hazard rates of the prepaid and defaulted portions of the portfolio.

Appendix A:

Since $X(s)=\int_t^s W(u)du$, then we have $X(s)=[R(s), \quad X_H(s)]$, where $R(s)=\int_t^s r(u)du$, denoted as the cumulative default-free short interest rates; $X_H(s)=\int_t^s r_H(u)du$, denoted as the cumulative housing return. Using the method shown in Liao et al. (2008), we have:

$$\mu_X(s)=[\mu_R(s) \quad \mu_{X_H}(s)]' ; \quad \mu_W(s)=[\mu_r(s) \quad \mu_{r_H}(s)]' ;$$

$$V_X(s)=\begin{bmatrix} V_R(s) & V_{R,X_H}(s) \\ V_{R,X_H}(s) & V_{X_H}(s) \end{bmatrix}; \quad V_W(s)=\begin{bmatrix} V_r(s) & V_{r,r_H}(s) \\ V_{r,r_H}(s) & V_{r_H}(s) \end{bmatrix}; \text{ and}$$

$$V_{X,W}(s)=\begin{bmatrix} V_{r,R}(s) & V_{R,r_H}(s) \\ V_{r,X_H}(s) & V_{X_H,r_H}(s) \end{bmatrix},$$

where

$$\mu_r(s) = f(s)+\frac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2, \quad f(s) \text{ is the forward rate for time } s \text{ at time } t,$$

$$\mu_{r_H}(s)=\bar{r}_H(t)+(r_H(t)-\bar{r}_H(t))e^{-a_H(s-t)} ;$$

$$\mu_R(s) = f(s)(s-t)+\frac{\sigma_r^2}{2a^2}((s-t)-\frac{2}{a}(1-e^{-a(s-t)})+\frac{1}{2a}(1-e^{-2a(s-t)})) ;$$

$$\mu_{X_H}(s)=\bar{r}_H(t)(s-t)+(r_H(t)-\bar{r}_H(t))\,(\frac{1-e^{-a_H(s-t)}}{a_H}) ;$$

$$V_r(s) = \frac{\sigma_r^2}{2a}(1-e^{-2a(s-t)}) ;$$

$$V_{r_H}(s) = \frac{\sigma_H^2}{2a_H}(1-e^{-2a_H(s-t)}) ;$$

$$V_R(s) = \frac{\sigma_r^2}{a^2}((s-t)-\frac{2}{a}(1-e^{-a(s-t)})+\frac{1}{2a}(1-e^{-2a(s-t)})) ;$$

$$V_{X_H}(s) = \frac{\sigma_H^2}{a_H^2}((s-t)-\frac{2}{a_H}(1-e^{-a_H(s-t)})+\frac{1}{2a_H}(1-e^{-2a_H(s-t)})) ;$$

$$V_{r,r_H}(s) = \sigma_H \sigma_r \phi_{rH} \left(\frac{1-e^{-(a+a_H)(s-t)}}{a+a_H}\right);$$

$$V_{R,X_H}(s) = \sigma_H \sigma_r \phi_{rH} \left((s-t) - \frac{1}{a}(1-e^{-a(s-t)}) - \frac{1}{a_H}(1-e^{-a_H(s-t)}) + \frac{1}{(a+a_H)}(1-e^{-(a+a_H)(s-t)}\right);$$

$$V_{r,R}(s) = \frac{\sigma_r^2}{2a^2}(1-e^{-a(s-t)})^2;$$

$$V_{X_H,r_H}(s) = \frac{\sigma_H^2}{2a_H^2}(1-e^{-a_H(s-t)})^2;$$

$$V_{r,X_H}(s) = \sigma_H \sigma_r \phi_{r,H} \left(\frac{1}{aa_H}(1-e^{-a(s-t)}) - \frac{1}{a_H(a+a_H)}(1-e^{-(a+a_H)(s-t)})\right); \text{ and}$$

$$V_{R,r_H}(s) = \sigma_H \sigma_r \phi_{r,H} \left(\frac{1}{aa_H}(1-e^{-a_H(s-t)}) - \frac{1}{a(a+a_H)}(1-e^{-(a+a_H)(s-t)})\right).$$

Appendix B

This Appendix presents the derivations of the formulas for the sensitivity analyses. Using the formula in Equation (23), we have the following results:

$$\frac{\partial \xi}{\partial \varpi} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T B(s)ds > 0\,;$$

$$\frac{\partial \xi}{\partial \rho} = -(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T Yt_{d_1}\Phi^r(s)\ \Phi(s)ds < 0\,;$$

$$\frac{\partial \xi}{\partial \kappa_1} = -(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T Yt_{d_1}\Phi^r(s)\ \Phi(s)ds < 0\,;$$

$$\frac{\partial \xi}{\partial t_{d_1}} = -(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T ((\rho+\kappa_1)Y\Phi^r(s) + \Phi^{r,\theta}(s)M(s))\Phi(s)ds < 0\,;$$

$$\frac{\partial \xi}{\partial \kappa_2} = -(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T (\varphi Y - Yt_{d_2}\Phi^r(s))\ \Phi(s)ds < 0\,;$$

$$\frac{\partial \xi}{\partial t_{d_2}} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T (\kappa_2 Y\Phi^r(s) > 0\,;$$

$$\frac{\partial \xi}{\partial \varphi} = -(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T \kappa_2 Y\ \Phi(s)ds < 0\,;$$

$$\frac{\partial \xi}{\partial \varpi_0} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T B(s)\Phi(s)ds > 0\,;$$

$$\frac{\partial \xi}{\partial \varpi_1} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T NB(s)\Phi(s)ds > 0\,;$$

$$\frac{\partial \xi}{\partial N} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T \varpi_1 B(s)\Phi(s)ds > 0\,;$$

$$\frac{\partial \xi}{\partial v_0} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T \Phi^{\pi}(s)M(s)\Phi(s)ds > 0\,;$$

$$\frac{\partial \xi}{\partial v_1} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T t_f\Phi^{\pi}(s)M(s)\Phi(s)ds > 0\,; \text{ and}$$

$$\frac{\partial \xi}{\partial t_f} = (\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T v_1\Phi^{\pi}(s)M(s)\Phi(s)ds > 0\,.$$

To analyze the influence of the model's parameter on the MSR fee, we let ϑ be the variables of $f(s)$, a, σ_r, $\bar{r}_H(t)$, $r_H(t)$, a_H, σ_H, and ϕ_{rH}. According to Appendix A, the partial derivatives of the elements of μ_X and μ_W with respect to $f(s)$ and $\bar{r}_H(t)$ are

$$\frac{\partial \mu_R(s)}{\partial f(s)} = (s-t),\quad \frac{\partial \mu_r(s)}{\partial f(s)} = 1,\quad \frac{\partial \mu_{X_H}(s)}{\partial \bar{r}_H(t)} = s - t - \frac{1-e^{-a_H(s-t)}}{a_H} \quad \text{and} \quad \frac{\mu_{r_H}(s)}{\partial \bar{r}_H(t)} = 1-e^{-a_H(s-t)}\,.$$

The following expressions show the partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to a and a_H:

$$\frac{\partial \mu_R(s)}{\partial a} = \sigma_r^2(-\frac{1}{a^3}(s-t)+\frac{3}{a^4}(1-e^{-a(s-t)})-\frac{1}{a^3}(e^{-a(s-t)}(s-t))$$
$$-\frac{3}{4a^4}(1-e^{-2a(s-t)})+\frac{1}{2a^3}(e^{-2a(s-t)}(s-t)));$$

$$\frac{\partial \mu_r(s)}{\partial a} = \frac{\sigma_r^2}{a^2}(1-e^{-a(s-t)})e^{-a(s-t)}(s-t)-\frac{\sigma_r^2}{a^3}(1-e^{-a(s-t)})^2;$$

$$\frac{\partial \mu_{X_H}(s)}{\partial a_H} = (r_h(t)-\bar{r}_h)\ (\frac{(s-t)e^{-a_H(u-t)}}{a_H}-\frac{1-e^{-a_H(s-t)}}{a_H^2});$$

$$\frac{\partial \mu_{r_H}(s)}{\partial a_H} = -(r_H(t)-\bar{r}_H(t))(s-t)e^{-a_H(s-t)};$$

$$\frac{\partial V_r(s)}{\partial a} = \frac{\sigma_r^2}{2a}(2(s-t)e^{-2a(s-t)}-\frac{1}{a}(1-e^{-2a(s-t)}));$$

$$\frac{\partial V_R(s)}{\partial a} = \sigma_r^2[-\frac{2}{a^3}(s-t)+\frac{6}{a^4}(1-e^{-a(s-t)})-\frac{2}{a^3}(e^{-a(s-t)}(s-t))$$
$$-\frac{3}{2a^4}(1-e^{-2a(s-t)})+\frac{1}{a^3}(e^{-2a(s-t)}(s-t))];$$

$$\frac{\partial V_{r,r_H}(s)}{\partial a} = \sigma_H\sigma_r\phi_{rH}\ \frac{1}{a+a_H}((s-t)e^{-(a+a_H)(s-t)}-\frac{1}{a+a_H}(1-e^{-(a+a_H)(s-t)}));$$

$$\frac{\partial V_{rX_H}(s)}{\partial a} = \sigma_H\sigma_r\phi_{r,H}(\frac{s-t}{aa_H}e^{-a(s-t)}-\frac{1}{a^2a_H}(1-e^{-a(s-t)})$$
$$+\frac{1}{a_H(a+a_H)^{-2}}(1-e^{-(a+a_H)(s-t)})\quad-\frac{s-t}{a_H(a+a_H)}e^{-(a+a_H)(s-t)});$$

$$\frac{\partial V_{rR}(s)}{\partial a} = \frac{\sigma_r^2}{a^2}(1-e^{-a(s-t)})e^{-a(s-t)}(s-t)-\frac{\sigma_r^2}{a^3}(1-e^{-a(s-t)})^2;$$

$$\frac{\partial V_{R,r_H}(t,s)}{\partial a} = \sigma_H\sigma_r\phi_{r,H}(\frac{-1}{a^2a_H}(1-e^{-a_H(s-t)})+\frac{1}{a(a+a_H)^{-2}}(1-e^{-(a+a_H)(s-t)})$$
$$+\frac{1}{a^{-2}(a+a_H)}(1-e^{-(a+a_H)(s-t)})-\frac{s-t}{a(a+a_H)}e^{-(a+a_H)(s-t)});$$

$$\frac{\partial V_{R,X_H}(s)}{\partial a} = \sigma_H\phi_{rH}\sigma_r(\frac{s-t}{a+a_H}e^{-2(a+a_H)(s-t)}-\frac{1}{2(a+a_H)^{-2}}(1-e^{-2(a+a_H)(s-t)}));$$

$$\frac{\partial V_{r_H}(s)}{\partial a_H} = \frac{\sigma_H^2}{2a_H}(2(s-t)e^{-2a_H(s-t)}-\frac{1}{a_H}(1-e^{-2a_H(s-t)}));$$

$$\frac{\partial V_R(s)}{\partial a_H} = \sigma_H^2[-\frac{2}{a_H^3}(s-t)+\frac{6}{a_H^4}(1-e^{-a_H(s-t)})-\frac{2}{a_H^3}(e^{-a_H(s-t)}(s-t))$$
$$-\frac{3}{2a_H^4}(1-e^{-2a_H(s-t)})+\frac{1}{a_H^3}(e^{-2a_H(s-t)}(s-t))];$$

$$\frac{\partial V_{r,r_H}(s)}{\partial a_H} = \sigma_H \sigma_r \phi_{rH} \ \frac{1}{a+a_H}((s-t)e^{-(a+a_H)(s-t)} - \frac{1}{a+a_H}(1-e^{-(a+a_H)(s-t)})) \ ;$$

$$\frac{\partial V_{rX_H}(s)}{\partial a_H} = \sigma_H \sigma_r \phi_{r,H}(\frac{-1}{aa_H^2}(1-e^{-a(s-t)}) + \frac{1}{a_H(a+a_H)^{-2}}(1-e^{-(a+a_H)(s-t)})$$

$$+\frac{1}{a_H^{-2}(a+a_H)}(1-e^{-(a+a_H)(s-t)}) - \frac{s-t}{a_H(a+a_H)}e^{-(a+a_H)(s-t)}) \ ;$$

$$\frac{\partial V_{r_H,X_H}(s)}{\partial a_H} = \frac{\sigma_H^2}{a_H^2}(1-e^{-a_H(s-t)})e^{-a_H(s-t)}(s-t) - \frac{\sigma_H^2}{a_H^3}(1-e^{-a_H(s-t)})^2 \ ;$$

$$\frac{\partial V_{R,r_H}(s)}{\partial a_H} = \sigma_H \sigma_r \phi_{r,H}(\frac{s-t}{aa_H}e^{-a_H(s-t)} - \frac{1}{a_H^2 a}(1-e^{-a_H(s-t)})$$

$$+\frac{1}{a(a+a_H)^{-2}}(1-e^{-(a+a_H)(s-t)}) - \frac{s-t}{a(a+a_H)}e^{-(a+a_H)(s-t)}) \ ; \text{ and}$$

$$\frac{\partial V_{R,X_H}(s)}{\partial a_H} = \sigma_H \phi_{rH} \sigma_r (\frac{s-t}{a+a_H}e^{-2(a+a_H)(s-t)} - \frac{1}{2(a+a_H)^{-2}}(1-e^{-2(a+a_H)(s-t)})).$$

The partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to σ_r are

$$\frac{\partial \mu_R(s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}((s-t) - \frac{2}{a}(1-e^{-a(s-t)}) + \frac{1}{2a}(1-e^{-2a(s-t)})) \ ,$$

$$\frac{\partial \mu_r(s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})^2 ,$$

$$\frac{\partial V_R(s)}{\partial \sigma_r} = \frac{2\sigma_r}{a^2}[(s-t) - \frac{2}{a}(1-e^{-a(s-t)}) + \frac{1}{2a}(1-e^{-2a(s-t)})] \ ,$$

$$\frac{\partial V_{r,R}(s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}(1-e^{-a(s-t)})^2 ,$$

$$\frac{\partial V_{R,X_H}(s)}{\partial \sigma_r} = \sigma_H \phi_{rH} \ \frac{1}{2(a+a_H)}(1-e^{-2(a+a_H)(s-t)}) \ ,$$

$$\frac{\partial V_{r,X_H}(s)}{\partial \sigma_r} = \sigma_H \phi_{r,H}(\frac{1}{aa_H}(1-e^{-a(s-t)}) - \frac{1}{a_H(a+a_H)}(1-e^{-(a+a_H)(s-t)})) \ ,$$

$$\frac{\partial V_{R,r_H}(s)}{\partial \sigma_r} = \sigma_H \phi_{r,H}(\frac{1}{aa_H}(1-e^{-a_H(s-t)}) - \frac{1}{a(a+a_H)}(1-e^{-(a+a_H)(s-t)})) \ , \text{ and}$$

$$\frac{\partial V_{r,r_H}(s)}{\partial \sigma_r} = \sigma_H \phi_{rH} \ (\frac{1-e^{-(a+a_H)(u-t)}}{a+a_H}) \ .$$

The partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to σ_{X_H} are

$$\frac{\partial \mu_{r_H}(s)}{\partial \sigma_H} = 0 \ ,$$

$$\frac{\partial \mu_{X_H}(s)}{\partial \sigma_H} = 0,$$

$$\frac{\partial V_{r_H}(s)}{\partial \sigma_H} = \frac{2\sigma_H}{2a_H}(1-e^{-2a_H(s-t)}),$$

$$\frac{\partial V_{X_H}(s)}{\partial \sigma_H} = \frac{2\sigma_H}{a_H^2}((s-t)-\frac{2}{a_H}(1-e^{-a_H(s-t)})+\frac{1}{2a_H}(1-e^{-2a_H(s-t)})),$$

$$\frac{\partial V_{R,X_H}(s)}{\partial \sigma_H} = \sigma_r \phi_{rH}\ \frac{1}{2(a+a_H)}(1-e^{-2(a+a_H)(s-t)}),$$

$$\frac{\partial V_{r,X_H}(s)}{\partial \sigma_H} = \sigma_r \phi_{r,H}(\frac{1}{aa_H}(1-e^{-a(s-t)})-\frac{1}{a_H(a+a_H)}(1-e^{-(a+a_H)(s-t)})),$$

$$\frac{\partial V_{R,r_H}(s)}{\partial \sigma_H} = \sigma_r \phi_{r,H}(\frac{1}{aa_H}(1-e^{-a_H(s-t)})-\frac{1}{a(a+a_H)}(1-e^{-(a+a_H)(s-t)})),$$

$$\frac{\partial V_{r,r_H}(s)}{\partial \sigma_H} = \sigma_r \phi_{rH}\ (\frac{1-e^{-(a+a_H)(u-t)}}{a+a_H}),\text{ and}$$

$$\frac{\partial V_{X_H,r_H}(s)}{\partial \sigma_H} = \frac{2\sigma_H}{2a_H^2}(1-e^{-a_H(s-t)})^2.$$

Furthermore, the partial derivatives of the elements of μ_X, μ_W, Ω_X and Ω_{XW} with respect to ϕ_{rH} are as follows:

$$\frac{\partial V_{R,X_H}(s)}{\partial \phi_{r,H}} = \sigma_H \sigma_r\ \frac{1}{2(a+a_H)}(1-e^{-2(a+a_H)(s-t)}),$$

$$\frac{\partial V_{r,X_H}(s)}{\partial \phi_{r,H}} = \sigma_H \sigma_r(\frac{1}{aa_H}(1-e^{-a(s-t)})-\frac{1}{a_H(a+a_H)}(1-e^{-(a+a_H)(s-t)})),$$

$$\frac{\partial V_{R,r_H}(s)}{\partial \phi_{r,H}} = \sigma_H \sigma_r(\frac{1}{aa_H}(1-e^{-a_H(s-t)})-\frac{1}{a(a+a_H)}(1-e^{-(a+a_H)(s-t)})),\text{ and}$$

$$\frac{\partial V_{r,r_H}(s)}{\partial \phi_{r,H}} = \sigma_H \sigma_r\ (\frac{1-e^{-(a+a_H)(u-t)}}{a+a_H}).$$

According to the above results, we then have the following expressions:

$$\frac{\partial \xi}{\partial \vartheta} = -\xi(\int_t^T M(s)\Phi(s)ds)^{-1} \times \int_t^T M(s)\frac{\partial \Phi(s)}{\partial \vartheta}ds$$

$$+(\int_t^T M(s)\Phi(s)ds)^{-1}$$

$$\times[(\int_t^T (\varpi_0+\varpi_1 N)\frac{\partial B(s)}{\partial\vartheta}-((\rho+\kappa_1)Yt_{d_1}\Phi^r(s)+(\varphi\kappa_2 Y-\kappa_2 Yt_{d_2}\Phi^r(s))$$

$$-(v_0+v_1 t_f)\Phi^\pi(s)M(s)+\Phi^{r,\theta}(s)M(s))\frac{\partial\Phi(s)}{\partial\vartheta}ds)$$

$$+\int_t^T(-((\rho+\kappa_1)Yt_{d_1}\frac{\partial\Phi^r(s)}{\partial\vartheta}-\kappa_2 Yt_{d_2}\frac{\partial\Phi^r(s)}{\partial\vartheta}$$

$$-(v_0+v_1 t_f)\frac{\partial\Phi^\pi(s)}{\partial\vartheta}M(s)+\frac{\partial\Phi^{r,\theta}(s)}{\partial\vartheta}t_{d_1}M(s))\Phi(s)ds)],$$

where

$$\frac{\partial B(s)}{\partial\vartheta}=(-\frac{\partial\mu_R(s)}{\partial\vartheta}+\frac{1}{2}\frac{\partial V_R(s)}{\partial\vartheta})B(s),$$

$$\frac{\partial\Phi(s)}{\partial\vartheta}=(-A_E'\frac{\partial\mu_X(s)}{\partial\vartheta}+\frac{1}{2}A_E'\frac{\partial V_X(s)}{\partial\vartheta}A_E)\Phi(s),$$

$$\frac{\partial\Phi^r(s)}{\partial\vartheta}=-A_r'\frac{\partial\mu_W(s)}{\partial\vartheta}+A_E'\frac{\partial V_{XW}(s)}{\partial\vartheta}A_r,$$

$$\frac{\partial\Phi^\pi(s)}{\partial\vartheta}=-A_\pi'\frac{\partial\mu_W(s)}{\partial\vartheta}+A_E'\frac{\partial V_{XW}(s)}{\partial\vartheta}A_\pi, \text{ and}$$

$$\frac{\partial\Phi^{r,\theta}(s)}{\partial\vartheta}=a_0^\rho(s)(A_r'\frac{\partial\mu_W(s)}{\partial\vartheta}-A_E'\frac{\partial V_{XW}(s)}{\partial\vartheta}A_r)$$

$$+(A_r'\frac{\partial\mu_W(s)}{\partial\vartheta}-A_E'\frac{\partial V_{XW}(s)}{\partial\vartheta}A_r)\,(A_\theta'\frac{\partial\mu_W(s)}{\partial\vartheta}-A_E'\frac{\partial V_{XW}(s)}{\partial\vartheta}A_\theta)+A_r'\frac{\partial V_W(s)}{\partial\vartheta}A_\theta.$$

Regarding the influence of the baseline prepayment and default hazard rate, we have the following:

$$\frac{\partial\xi}{\partial a_0^\theta}=-\xi(\int_t^T M(s)\Phi(s)ds)^{-1}\times\int_t^T M(s)\frac{\partial\Phi(s)}{\partial a_0^\theta}ds$$

$$+(\int_t^T M(s)\Phi(s)ds)^{-1}\times[(\int_t^T -((\rho+\kappa_1)Yt_{d_1}\Phi^r(s)+(\varphi\kappa_2 Y-\kappa_2 Yt_{d_2}\Phi^r(s))$$

$$-(v_0+v_1 t_f)\Phi^\pi(s)M(s)+\Phi^{r,\theta}(s)M(s))\frac{\partial\Phi(s)}{\partial a_0^\theta}ds)$$

$$-\int_t^T\frac{\partial\Phi^{r,\theta}(s)}{\partial a_0^\theta}M(s))\Phi(s)ds]>0,$$

$$\frac{\partial\xi}{\partial a_0^\pi}=-\xi(\int_t^T M(s)\Phi(s)ds)^{-1}\times\int_t^T M(s)\frac{\partial\Phi(s)}{\partial a_0^\pi}ds$$

$$+\left(\int_t^T M(s)\Phi(s)ds\right)^{-1} \times [(\int_t^T -((\rho+\kappa_1)Yt_{d_1}\Phi^r(s)+(\varphi\kappa_2 Y-\kappa_2 Yt_{d_2}\Phi^r(s))$$

$$-(v_0+v_1t_f)\Phi^\pi(s)M(s)+\Phi^{r,\theta}(s)M(s))\frac{\partial\Phi(s)}{\partial a_0^\pi}ds)$$

$$+\int_t^T (v_0+v_1t_f)\frac{\partial\Phi^\pi(s)}{\partial a_0^\pi}M(s))\Phi(s)ds)>0,$$

where

$\dfrac{\partial\Phi(s)}{\partial a_0^\theta}=-(s-t)\Phi(s)$; $\dfrac{\partial\Phi(s)}{\partial a_0^\pi}=-(s-t)\Phi(s)$; $\dfrac{\partial\Phi^\pi(s)}{\partial a_0^\pi}=1$; and

$\dfrac{\partial\Phi^{r,\theta}(s)}{\partial a_0^\theta}=(A_r'\mu_W(s)-A_E'V_{XW}(s)A_r)$.